MAULDIN-WILLIAMS GRAPHS, MORITA EQUIVALENCE AND ISOMORPHISMS

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Abstract. We describe a method for associating some non-self-adjoint algebras to Mauldin-Williams graphs and we study the Morita equivalence and isomorphism of these algebras.

We also investigate the relationship between the Morita equivalence and isomorphism class of the C^*-correspondences associated with Mauldin-Williams graphs and the dynamical properties of the Mauldin-Williams graphs.

1. Introduction

In this note we follow the notation from [8]. By a *Mauldin-Williams graph* (see [14]), we mean a system $\mathcal{G} = (G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E})$, where $G = (V, E, r, s)$ is a graph with a finite set of vertices V, a finite set of edges E, a *range* map r and a *source* map s, and where $\{T_v, \rho_v\}_{v\in V}$ and $\{\phi_e\}_{e\in E}$ are families such that:

(1) Each T_v is a compact metric space with a prescribed metric ρ_v, $v \in V$.
(2) For $e \in E$, ϕ_e is a continuous map from $T_{r(e)}$ to $T_{s(e)}$ such that

$$c_1\rho_{r(e)}(x, y) \le \rho_{s(e)}(\phi_e(x), \phi_e(y)) \le c\rho_{r(e)}(x, y)$$

for some constants c_1, c satisfying $0 < c_1 \le c < 1$ (independent of e) and all $x, y \in T_{r(e)}$.

We shall assume, too, that the source map s and the range map r are surjective. Thus, we assume that there are no sinks and no sources in the graph G.

In [8] we associated to a Mauldin-Williams graph $\mathcal{G} = (G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E})$ a so-called C^*-*correspondence* $\mathcal{X}$ over the C^*-algebra $A = C(T)$, where $T = \coprod_{v\in V} T_v$ is the disjoint union of the spaces $T_v, v \in V$, as follows. Let $E \times_G T = \{(e, x) \,|\, x \in T_{r(e)}\}$. Then, by our finiteness assumptions, $E \times_G T$ is a compact space. We set $\mathcal{X} = C(E \times_G T)$ and view $\mathcal{X}$ as a C^*-correspondence over $C(T)$ via the formulae:

$$\begin{aligned} \xi \cdot a(e, x) &:= \xi(e, x)a(x), \\ a \cdot \xi(e, x) &:= a \circ \phi_e(x)\xi(e, x) \end{aligned}$$

and

$$\langle \xi, \eta\rangle_A(x) := \sum_{\substack{e\in E \\ x\in T_{r(e)}}} \overline{\xi(e, x)}\eta(e, x),$$

where $a \in C(T)$ and $\xi, \eta \in C(E \times_G T)$. With these data we can form the tensor algebra $\mathcal{T}_+(\mathcal{X})$ as prescribed in [15] and [16]. Our main result is:

Theorem 1.1. *For $i = 1, 2$, let $\mathcal{G}_i = (G_i, (K^i_v)_{v\in V_i}, (\phi^i_e)_{e\in E_i})$ be two Mauldin-Williams graphs. Let $A_i = C(K^i)$ and let $\mathcal{X}_i$ be the associated C^*-algebras and C^*-correspondences. Then the following are equivalent:*

(1) $\mathcal{T}_+(\mathcal{X}_1)$ *is strongly Morita equivalent to* $\mathcal{T}_+(\mathcal{X}_2)$ *in the sense of* [2].
(2) $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are strongly Morita equivalent in the sense of* [16].
(3) $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are isomorphic as* C^**-correspondeces.*
(4) $\mathcal{T}_+(\mathcal{X}_1)$ *is completely isometrically isomorphic to* $\mathcal{T}_+(\mathcal{X}_2)$.

We find this result especially remarkable in light of Theorem 2.3 from [8, Theorem 1.1] (see also Section 4.2 from [18]), which states that the Cuntz-Pimsner algebra, $\mathcal{O}(\mathcal{X})$, which is the C^*-envelope of the tensor algebra $\mathcal{T}_+(\mathcal{X})$, depends only of the structure of the underlying graph. In particular, our results lead to examples of different non-self-adjoint algebras which are not completely isometrically isomorphic, but have the same C^*-envelope, namely $\mathcal{O}_n$.

To understand further the relationship between the tensor algebra and the Mauldin-Williams graph, we study the isomorphism class of our C^*-correspondences and tensor algebras in terms of the dynamics of the Mauldin-Williams graph. Roughly, we find that two C^*-correspondences associated to two Mauldin-Williams graphs, $(G_i, (K_v^{(i)})_{v\in V_i}, (\phi_e^i)_{e\in E_i})$, $i = 1, 2$ are isomorphic if the maps ϕ_e^1 and ϕ_e^2 are locally conjugate in a sense that will be made precise later.

2. Non-self-adjoint Algebras Associated with Mauldin-Williams Graphs

Definition 2.1. An *invariant list* associated with a Mauldin-Williams graph $\mathcal{G} = (G, \{T_v, \rho_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ is a family $(K_v)_{v\in V}$ of compact sets, such that $K_v \subset T_v$, for all $v \in V$ and such that

$$K_v = \bigcup_{e\in E, s(e)=v} \phi_e(K_{r(e)}).$$

Since each ϕ_e is a proper contraction, $\mathcal{G}$ has a unique invariant list (see [14, Theorem 1]). We set $T := \bigcup_{v\in V} T_v$ and $K := \bigcup_{v\in V} K_v$ and we call K the *invariant set* of the Mauldin-Williams graph.

In the particular case when we have one vertex v and n edges, i.e. in the setting of an *iterated function system,* the invariant set is the unique compact subset $K := K_v$ of $T = T_v$ such that

$$K = \phi_1(K) \cup \cdots \cup \phi_n(K).$$

Note that the $*$-homomorphism $\Phi : A \to \mathcal{L}(\mathcal{X})$, $(\Phi(a)\xi)(e, x) = a \circ \phi_e(x)\xi(e, x)$, which gives the left action of the C^*-correspondence associated to a Mauldin-Williams graph, is faithful if and only if $K = T$. In this note we assume that T equals the invariant set K.

Kajiwara and Watatani have proved in [10, Lemma 2.3] that, if the contractions are proper, the invariant set of an iterated function system has no isolated point. Their proof can be easily generalized to the invariant set of a Mauldin-Williams graph. Hence K has no isolated points.

For a C^*-correspondence $\mathcal{X}$ over a C^*-algebra A, the (full) *Fock space* over $\mathcal{X}$ is

$$\mathcal{F}(\mathcal{X}) = A \oplus \mathcal{X} \oplus \mathcal{X}^{\otimes 2} \oplus \cdots .$$

We write Φ_∞ for the left action of A on $\mathcal{F}(\mathcal{X})$, $\Phi_\infty(a) = \mathrm{diag}(a, \Phi^{(1)}(a), \Phi^{(2)}(a), \cdots)$, where $\Phi^{(n)}$ is the left action of A on $\mathcal{X}^{\otimes n}$ ($\Phi^{(1)} = \Phi$, the left action of A on $\mathcal{X}$). For $\xi \in \mathcal{X}$, the creation operator determined by ξ is defined by the formula $T_\xi(\eta) = \xi \otimes \eta$, for all $\eta \in \mathcal{F}(\mathcal{X})$.

Definition 2.2. The *tensor algebra* of $\mathcal{X}$, denoted by $\mathcal{T}_+(\mathcal{X})$, is the norm closed subalgebra of $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ generated by $\Phi_\infty(A)$ and the creation operators T_ξ, for $\xi \in \mathcal{X}$ (see [15] and [16]). The C^*-algebra generated by $\mathcal{T}_+(\mathcal{X})$ is denoted by $\mathcal{T}(\mathcal{X})$ and it is called the *Toeplitz algebra* of the C^*-correspondence $\mathcal{X}$.

We may regard each finite sum $\sum_{n=0}^{N}\mathcal{X}^{\otimes n}$ as a subspace of $\mathcal{F}(\mathcal{X})$ and we may regard $\mathcal{L}(\sum_{n=0}^{N}\mathcal{X}^{\otimes n})$ as a subalgebra of $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ in the obvious way. Let B be the C^*-subalgebra of $\mathcal{L}(\mathcal{F}(\mathcal{X}))$ generated by all the $\mathcal{L}(\sum_{n=0}^{N}\mathcal{X}^{\otimes n})$ as N ranges over the non-negative integers. Then $\mathcal{T}(\mathcal{X}) \subset M(B)$, the multiplier algebra of B. The Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{X})$ is defined to be the image of $\mathcal{T}(\mathcal{X})$ in the corona algebra $M(B)/B$ (see [15] and [17]).

By a *homomorphism* from an $A_1 - B_1$ C^*-correspondence $\mathcal{X}_1$, *to an* $A_2 - B_2$ C^**-correspondence* $\mathcal{X}_2$ we mean a triple (α, V, β), where $\alpha : A_1 \to A_2$, $\beta : B_1 \to B_2$ are C^*-homomorphisms and $V : \mathcal{X}_1 \to \mathcal{X}_2$ is a linear map such that $V(a\xi b) = \alpha(a)V(\xi)\beta(b)$ and such that $\langle V(\xi), V(\eta)\rangle_{B_2} = \beta(\langle \xi, \eta\rangle_{B_1})$ (see [16, Section 1]). When $A_1 = A_2$ and $B_1 = B_2$, we will consider $\alpha \in \mathrm{Aut}(A_1)$ and $\beta \in \mathrm{Aut}(B_1)$. This, then, forces V to be isometric. If V is also surjective, we shall say that V is a *correspondence isomorphism over* (α, β). If, moreover, $A_1 = B_1$ and $\alpha = \beta$, we say that V is a *correspondence isomorphism over* α.

A central concept for our work in this note is the *strong Morita equivalence* for C^*-correspondences defined in [16, Definition 2.1], which we review here.

Definition 2.3. If $\mathcal{X}$ is a C^*-correspondence over a C^*-algebra A, and $\mathcal{Y}$ is a C^*-correspondence over a C^*-algebra B, we say that $\mathcal{X}$ and $\mathcal{Y}$ are *strongly Morita equivalent* if A and B are strongly Morita equivalent via an A-B equivalence bimodule $\mathcal{Z}$ (in which case we write $A \overset{\mathrm{SME}}{\sim}_{\mathcal{Z}} B$), for which there is an A-B correspondence isomorphism (id, W, id) from $\mathcal{Z} \otimes_B \mathcal{Y}$ onto $\mathcal{X} \otimes_A \mathcal{Z}$. This means, in particular, that $W(a\xi b) = aW(\xi)b$ for all $a \in A, b \in B$ and $\xi \in \mathcal{Z} \otimes_B \mathcal{Y}$ and that $\langle W(\xi), W(\eta)\rangle_B = \langle \xi, \eta\rangle_B$.

We say that a C^*-correspondence $\mathcal{X}$ over a C^*-algebra A is *aperiodic* if: for all $n \geq 1$, for all $\xi \in \mathcal{X}^{\otimes n}$ and for all hereditary subalgebras $B \subseteq A$, we have

$$\inf\left\{\left\|\Phi^{(n)}(a)\xi a\right\| \mid a \geq 0, a \in B, \|a\| = 1\right\} = 0.$$

It was proved in [16, Theorem 3.2, Theorem 3.5] that if $\mathcal{X}$ and $\mathcal{Y}$ are strongly Morita equivalent, then $\mathcal{T}_+(\mathcal{X})$ and $\mathcal{T}_+(\mathcal{Y})$ (respectively $\mathcal{T}(\mathcal{X})$ and $\mathcal{T}(\mathcal{Y})$, $\mathcal{O}(\mathcal{X})$ and $\mathcal{O}(\mathcal{Y})$) are strongly Morita equivalent. Also, if $\mathcal{X}$ and $\mathcal{Y}$ are aperiodic C^*-correspondences over the C^*-algebras A and B, respectively, and if $\mathcal{T}_+(\mathcal{X})$ and $\mathcal{T}_+(\mathcal{Y})$ are strongly Morita equivalent in the sense of [2], then $\mathcal{X}$ and $\mathcal{Y}$ are strongly Morita equivalent (see [16, Theorem 7.2]).

To study the aperiodicity and strong Morita equivalence of C^*-correspondences associated to Mauldin-Williams graphs, we need the following lemma which gives an equivalent description of when a C^*-correspondence is aperiodic.

Lemma 2.4. *(*[16, Lemma 5.2]*). The* C^**-correspondence* $\mathcal{X}$ *is aperiodic if and only if given* $a_0 \in A$, $a_0 \geq 0$, $\xi^k \in \mathcal{X}^{\otimes k}$, $1 \leq k \leq n$ *and* $\varepsilon > 0$, *there is an* x *in the hereditary subalgebra* $\overline{a_0 A a_0}$, *with* $x \geq 0$ *and* $\|x\| = 1$, *such that*

$$\|xa_0x\| > \|a_0\| - \varepsilon$$

and

$$\|\Phi^{(k)}(x)\xi^k x\| < \varepsilon \text{ for } 1 \leq k \leq n.$$

For a directed graph $G = (V, E, r, s)$ and for $k \geq 2$, we define

$$E^k := \{\alpha = (\alpha_1, \cdots, \alpha_k) \ : \ \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}), i = 1, \cdots, k-1\}$$

to be the set of paths of length k in the graph G. We define also the infinite path space to be

$$E^{\infty} := \{(\alpha_i)_{i\in\mathbb{N}} \;:\; \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for all } i \in \mathbb{N}\}$$

For $\alpha \in E^k$, we write $\phi_\alpha = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_k}$.

Proposition 2.5. *Let $\mathcal{G} = (G, (K_v)_{v\in V}, (\phi_e)_{e\in E})$ be a Mauldin-Williams graph with the invariant set K. Let $A = C(K)$ be the associated C^*-algebra and let $\mathcal{X}$ be the associated C^*-correspondence. Then the C^*-correspondence $\mathcal{X}$ is aperiodic.*

Proof. Note that $\phi_\alpha : K_{r(\alpha)} \to K_{s(\alpha)}$, with $\alpha \in E^k$ and $k \in \mathbb{N}$, has a fixed point if and only if $r(\alpha) = s(\alpha)$, i.e. α is a cycle in the graph G.

Fix $n_0 \in \mathbb{N}$, choose $k \in \mathbb{N}, 1 \le k \le n_0$; let $a_0 \in A$ with $a_0 \ge 0$; let $\xi^k \in \mathcal{X}^{\otimes k}$ and let $\varepsilon > 0$. We verify the criterion in Lemma 2.4 first when $n_0 = k = 1$.

Without loss of generality, we assume that $\|a_0\| = 1$. Then we can find $t_0 \in K$ such that $|a_0(t_0)| \ge 1 - \varepsilon$ and t_0 is not a fixed point for any ϕ_e , $e \in E$. Let $v_0 \in V$ be such that $t_0 \in K_{v_0}$. Choose $\delta_1 > 0$ such that $B(t_0, \delta_1) \subset K_{v_0}$ and $B(\phi_e(t_0), \delta_1) \cap B(t_0, \delta_1) = \emptyset$ for all $e \in E$ for which $r(e) = v_0$. Let

$$\delta_2 := \begin{cases} \min\{\rho_{v_0}(t_0, t) \mid a_0(t) = 0\}, & \text{if } \{t \in K_{v_0} \;:\; a_0(t) = 0\} \ne \emptyset \\ \delta_1, & \text{otherwise.} \end{cases}$$

Set $\delta = \min\{\delta_1, \delta_2\}$ and let $x \in A$, $x \ge 0$ be such that

$$x(t) = \begin{cases} 1, & \text{if} \quad t = t_0 \\ 0, & \text{if} \quad t \in K \setminus B(t_0, \delta). \end{cases}$$

Since $x(t) > 0$ only when $a_0(t) > 0$, it follows that $x \in \overline{a_0 A a_0}$. Moreover $x(t_0)a_0(t_0)x(t_0) > 1 - \varepsilon$, hence $\|x a_0 x\| > 1 - \varepsilon$.

Fix $t \in K$. If $t \in B(t_0, \delta)$ then $\phi_e(t) \notin B(t_0, \delta)$, by our choice of δ_1 and the fact that each map ϕ_e is a contraction, for all $e \in E$ such that $r(e) = v_0$; so $x \circ \phi_e(t)x(t) = 0$. If $t \notin B(t_0, \delta)$, then $x(t) = 0$, hence $x \circ \phi_e(t)x(t) = 0$, for all $e \in E$ such that $t \in K_{r(e)}$. Therefore $(\Phi(x)\xi x)(e, t) = x \circ \phi_e(t)\xi(e, t)x(t) = 0$ for all $(e, t) \in E \times_G K$. Since

$$\langle \Phi(x)\xi x, \Phi(x)\xi x\rangle_A(t) = \sum_{\substack{e\in E \\ t\in K_{r(e)}}} (x \circ \phi_e(t))^2 \mid \xi(e, t) \mid^2 x(t)^2,$$

we see that $\|\Phi(x)\xi x\| = 0$.

For $n_0 = 2$, we choose $t_0 \in K$ such that $a_0(t_0) > 1 - \varepsilon$ and t_0 is not a fixed point for any ϕ_α with $\alpha \in E^2$. Let $v_0 \in V$ be such that $t_0 \in K_{v_0}$. Let $\delta_1 > 0$ be such that $B(\phi_\alpha(t_0), \delta_1) \cap B(t_0, \delta_1) = \emptyset$, for all $\alpha \in E^2$ for which $r(\alpha) = v_0$, and such that $B(t_0, \delta_1) \subset K_{v_0}$. Choosing δ_2, δ and x as before, we conclude that $x \in \overline{a_0 A a_0}$ and $\|x\| > 1 - \varepsilon$. Moreover, we have $x \circ \phi_\alpha(t)x(t) = 0$ for all $t \in K$, $\alpha \in E \cup E^2$ (since ϕ_α is a contraction, for all $\alpha \in E \cup E^2$); and since

$$\left\langle \Phi^{(2)}(x)\xi^2 x, \Phi^{(2)}(x)\xi^2 x\right\rangle_A(t) = \sum_{\substack{\alpha\in E^2 \\ t\in K_{r(\alpha)}}} (x \circ \phi_\alpha(t))^2 \left|\xi_2^2(\alpha_2, t)\right|^2 \left|\xi_1^2(\alpha_1, \phi_{\alpha_1}(t))\right|^2 x(t)^2 = 0,$$

it follows that $\left\|\Phi^{(k)}(x)\xi^k x\right\| = 0$ for $k = 1, 2$. Applying the same argument inductively, we see that $\mathcal{X}$ is an aperiodic C^*-correspondence. ∎

Let K^1 and K^2 be two compact metric spaces. Let $A_1 = C(K^1)$ and $A_2 = C(K^2)$. If $A_1 \overset{\mathrm{SME}}{\sim}_{\mathcal{Z}} A_2$, then the Rieffel correspondence determines a unique homeomorphism $f : K^1 \to K^2$ and a unique Hermitian line bundle $\mathcal{L}$ over $\mathrm{Graph}(f) = \{(x, f(x)) \,:\, x \in K^1\}$, such that $\mathcal{Z}$ is isomorphic to $\Gamma(\mathcal{L})$ (see [21], [20, Section 3.3 and Example 4.55], [19, Appendix (A)]), where $\Gamma(\mathcal{L})$ is the imprimitivity bimodule of the cross sections of $\mathcal{L}$ endowed with the following structure:

$$\begin{aligned}(a \cdot s \cdot b)(x, f(x)) &= a(x)s(x, f(x))b(f(x), \\ \langle s_1, s_2\rangle_{A_2}(y) &= \overline{s_1(f^{-1}(y), y)}s_2(f^{-1}(y), y), \\ \text{and } {}_{A_1}\langle s_1, s_2\rangle(x) &= s_1(x, f(x))\overline{s_2(x, f(x))},\end{aligned}$$

for all $a \in A_1$, $b \in A_2$, $s, s_1, s_2 \in \Gamma(\mathcal{L})$. We write $\mathcal{Z}(f, \mathcal{L})$ for $\Gamma(\mathcal{L})$.

We are ready to proof the main theorem.

Proof (*of Theorem* 1.1). By Proposition 2.5, $\mathcal{X}_1$ and $\mathcal{X}_2$ are aperiodic C^*-correspondences. Using [16, Theorem 7.2], we obtain that (1) implies (2).

Now we show that (2) implies (3). Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are strongly Morita equivalent. This implies that A_1 and A_2 are strongly Morita equivalent via an imprimitivity bimodule $\mathcal{Z}$ such that $\mathcal{Z} \otimes \mathcal{X}_2$ is isomorphic to $\mathcal{X}_1 \otimes \mathcal{Z}$. Let $f : K^1 \to K^2$ and $\mathcal{L}$ be the homeomorphism and the line bundle determined by the Rieffel correspondence. We have that $\mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2$ is isomorphic to $\mathcal{X}_1 \otimes \mathcal{Z}(f, \mathcal{L})$. Hence $\mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \widetilde{\mathcal{Z}(f, \mathcal{L})}$ is isomorphic to $\mathcal{X}_1$, where $\widetilde{\mathcal{Z}(f, \mathcal{L})}$ is the dual imprimitivity bimodule (see [20, Proposition 3.18]). We prove that $\mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \widetilde{\mathcal{Z}(f, \mathcal{L})}$ is isomorphic to $\mathcal{X}_2$ over an isomorphism α of A_1 and A_2.

Let $\alpha : A_1 \to A_2$ be defined by the formula $\alpha(a) = a \circ f^{-1}$ and let $V : \mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \widetilde{\mathcal{Z}(f, \mathcal{L})} \to \mathcal{X}_2$ be defined by the formula

$$V(s_1 \otimes \xi \otimes \widetilde{s_2})(e, y) = s_1(f^{-1}(\phi_e^2(y)), \phi_e^2(y))\xi(e, x)\overline{s_2(f^{-1}(y), y)}.$$

Then α is an isomorphism and

$$V(a \cdot s_1 \otimes \xi \otimes \widetilde{s_2} \cdot b) = a \cdot V(s_1 \otimes \xi \otimes \widetilde{s_2}) \cdot b,$$

for all $a, b \in A$, $s_1, s_2 \in \mathcal{Z}(f, \mathcal{L})$, $\xi \in \mathcal{X}_2$. Moreover we have that

$$\begin{aligned}&\langle V(s_1 \otimes \xi \otimes \widetilde{s_2}), V(t_1 \otimes \eta \otimes \widetilde{t_2}\rangle_{A_2}(y) \\ &= \sum_{\substack{e \in E \\ y \in K^2_{r(e)}}} \overline{V(s_1 \otimes \xi \otimes \widetilde{s_2})}(e, y)V(t_1 \otimes \eta \otimes \widetilde{t_2})(e, y) \\ &= \sum_{\substack{e \in E \\ y \in K^2_{r(e)}}} \Big(\overline{s_1(f^{-1}(\phi_e^2(y)), \phi_e^2(y))\xi(e, x)\overline{s_2(f^{-1}(y), y)}} \\ &\cdot t_1(f^{-1}(\phi_e^2(y)), \phi_e^2(y))\eta(e, x)\overline{t_2(f^{-1}(y), y)}\Big) \\ &= \langle s_1 \otimes \xi \otimes s_2, t_1 \otimes \eta \otimes t_2\rangle_{A_2},\end{aligned}$$

for all $s_1, s_2, t_1, t_2 \in \mathcal{Z}(f, \mathcal{L})$ and $\xi, \eta \in \mathcal{X}_2$. Also, for $\xi \in \mathcal{X}_2$, $V(1 \otimes \xi \otimes 1) = \xi$. Hence V is a correspondence isomorphism. Thus $\mathcal{X}_1$ is isomorphic to $\mathcal{X}_2$.

The rest is clear. □

It was shown in [8, Theorem 2.3] that the Cuntz-Pimsner algebra of the C^*-correspondence built from a Mauldin-Williams graph is isomorphic to the Cuntz-Krieger algebra of the underlying graph

$G = (V, E, r, s)$ (as defined in [12]). Hence, for C^*-correspondences associated to Mauldin-Williams graphs with the same underlying graph which are not isomorphic, we obtain tensor algebras which are not Morita equivalent, but have the same C^*-envelope, namely the Cuntz-Krieger algebra of the graph G.

3. The Isomorphism class of the C^*-Correspondences Associated With Mauldin Williams Graphs

In the following we analyze the relation between the isomorphism class of the C^*-correspondences associated with two Mauldin-Williams graphs, $\mathcal{G}_i = (G, (K_v^i)_{v\in V}, (\phi_e^i)_{e\in E})$, $i = 1, 2$ and the topological and dynamical properties of the Mauldin-Williams graphs.

Since, by [18, Section 4.2] and [8, Theorem 2.3], the Cuntz-Pimsner algebra associated to a Mauldin-Williams graph depends only on the structure of the underlying graph G, we will consider only Mauldin-Williams graphs having the same underlying graph $G = (V, E, r, s)$.

Next we determine necessary and sufficient conditions for the isomorphism of the C^*-correspondences associated to two Mauldin-Williams graphs.

Proposition 3.1. *For $i = 1, 2$, let $\mathcal{G}_i = (G, (K_v^i)_{v\in V}, (\phi_e^i)_{e\in E})$ be two Mauldin-Williams graphs over the same underlying graph G. Let $A_i = C(K^i)$, $i = 1, 2$, be the associated C^*-algebras and let $\mathcal{X}_i$, $i = 1, 2$, be the associated C^*-correspondences. If there is a homeomorphism $f : K^1 \to K^2$, a partition of open subsets $\{U_1, \dots, U_m\}$ for K^1, for some $m \in \mathbb{N}$, and if for each U_j there is a permutation $\sigma_j \in S_n$, where $n = |E|$, such that $f^{-1} \circ \phi^2_{\sigma_j(e)} \circ f|_{U_j} = \phi_e^1|_{U_j}$ and $f(K^1_{r(e)}) = K^2_{r(\sigma_j(e))}$ for all $e \in E$, $j \in \{1, \dots, m\}$, then $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic.*

Proof. Since f is a homeomorphism, the map $\beta : A_2 \to A_1$, defined by the equation $\beta(b) = b \circ f$ for all $b \in A_2$, is a C^*-isomorphism. Define $V : \mathcal{X}_2 \to \mathcal{X}_1$ by the formula

$$V(\xi)(e, x) = \sum_{k=1}^{m} \xi_{\sigma_k(e)}(f(x)) \cdot 1_{U_k}(x),$$

for all $(e, x) \in E \times_G K$, where $\xi_{\sigma_k(e)}(f(x)) := \xi(\sigma_k(e), f(x))$. We show that V is a C^*-correspondence isomorphism over β. Let $b_1, b_2 \in A_2$ and $\xi \in \mathcal{X}_2$. We have

$$\begin{aligned} V(b_1 \cdot \xi \cdot b_2)(e, x) &= \sum_{k=1}^{m} b_1 \circ \phi^2_{\sigma_k(e)}(f(x)) \xi_{\sigma_k(e)}(f(x)) b_2(f(x)) 1_{U_k}(x) \\ &= \sum_{k=1}^{m} b_1 \circ f \circ \phi_e^1(x) \xi_{\sigma_k(e)}(f(x)) 1_{U_k}(x) \cdot \beta(b_2)(x) \\ &= \beta(b_1) \cdot V(\xi) \cdot \beta(b_2)(e, x). \end{aligned}$$

Also

$$\langle V(\xi), V(\eta)\rangle_{A_1}(x) = \sum_{\substack{e\in E \\ f(x)\in K^2_{r(e)}}} \left(\sum_{k=1}^{m} \overline{\xi_{\sigma_k(e)}(f(x)} \eta_{\sigma_k(e)}(f(x)) 1_{U_k}(x) \right),$$

hence $\langle V(\xi), V(\eta)\rangle_{A_1} = \beta(\langle \xi, \eta\rangle_{A_2})$. Finally one can see that V is onto, hence V is a C^*-correspondence isomorphism. $\square$

Recall that, for $k \geq 2$, $E^k := \{\alpha = (\alpha_1, \cdots, \alpha_k) \ : \ \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}), i = 1, \cdots, k-1\}$, is the set of *paths of length* k in the graph G. Let $E^* = \bigcup_{k \in N} E^k$ be the space of finite paths in the graph G. Also the *infinite path space,* E^∞*,* is defined to be

$$E^\infty := \{(\alpha_i)_{i \in \mathbb{N}} \quad : \quad \alpha_i \in E \quad \text{and} \quad r(\alpha_i) = s(\alpha_{i+1}) \text{ for all } i \in \mathbb{N}\}.$$

For $v \in V$, we also define $E^k(v) := \{\alpha \in E^k \ : \ s(\alpha) = v\}$, and $E^*(v)$ and $E^\infty(v)$ are defined similarly. We consider $E^\infty(v)$ to be endowed with the metric: $\delta_v(\alpha, \beta) = c^{|\alpha \wedge \beta|}$ if $\alpha \neq \beta$ and 0 otherwise, where $\alpha \wedge \beta$ is the longest common prefix of α and β, and $|w|$ is the length of the word $w \in E^*$ (see [5, Page 116]). Then $E^\infty(v)$ is a compact metric space, and, since E^∞ equals the disjoint union of the spaces $E^\infty(v)$, E^∞ becomes a compact metric space in a natural way. Define the maps $\theta_e : E^\infty(r(e)) \to E^\infty(s(e))$ by the formula $\theta_e(\alpha) = e\alpha$, for all $\alpha \in E^\infty$ and for all $e \in E$. Then $(G, (E^\infty(v))_{v \in V}, (\theta_e)_{e \in E})$ is a Mauldin-Williams graph. We set $A_E := C(E^\infty)$ and we set $\mathcal{E}$ be the C^*-correspondence associated to this Mauldin-Williams graph. Let $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ be a Mauldin-Williams graph. For $(\alpha_1, \ldots, \alpha_n) \in E^n$ let $K_{(\alpha_1, \ldots, \alpha_n)} := \phi_{\alpha_1} \circ \ldots \phi_{\alpha_n}(K_{r(\alpha_n)})$. Then, for any infinite path $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in E^\infty$, $\bigcap_{n \geq 1} K_{(\alpha_1, \ldots, \alpha_n)}$ contains only one point. Therefore we can define a map $\pi : E^\infty \to K$ by $\{\pi(x)\} = \bigcap_{n \geq 1} K_{(\alpha_1, \ldots, \alpha_n)}$. Since $\pi(E^\infty)$ is also an invariant set, π is a continuous, onto map and $\pi(E^\infty(v)) = K_v$. Moreover, $\pi \circ \theta_e = \phi_e \circ \pi$.

We say that a Mauldin-Williams graph $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ is *totally disconnected* if $\phi_e(K_{r(e)}) \bigcap \phi_f(K_{r(f)}) = \emptyset$ if $s(e) = s(f)$ and $e \neq f$.

Corollary 3.2. *Let* $\mathcal{M} = (G, \{K_v, \rho_v\}_{v \in E^0}, \{\phi_e\}_{e \in E^1})$ *be a totally disconnected Mauldin-Williams graph. Let A be the C^*-algebra and $\mathcal{X}$ be the C^*-correspondence associated to this Mauldin-Williams graph. Then $\mathcal{X}$ is isomorphic with $\mathcal{E}$, as C^*-correspondences. In particular, one obtains that for any two totally disconnected Mauldin-Williams graphs having the same underlying graph G, the C^*-correspondences and tensor algebras associated are isomorphic.*

Proof. If the Mauldin-Williams graph is totally disconnected, then the map $\pi : E^\infty \to K$ defined above is a homeomorphism. Moreover $\pi \circ \theta_e \circ \pi^{-1} = \phi_e$ for all $e \in E$, therefore the associated C^*-correspondences are isomorphic. □

The converse of this corollary is true and will be proved later.

The next theorem is a converse of the Proposition 3.1. We note, however, that the family of open sets $\{U_i\}$ here is not required to be a partition of the compact set K^1, but only a finite open cover of it.

Theorem 3.3. *For $i = 1, 2$, let $\mathcal{G}_i = (G, (K_v^i)_{v \in V}, (\phi_e^i)_{e \in E})$ be two Mauldin-Williams graphs over the same underlying graph G. Let $A_i = C(K^i)$, $i = 1, 2$, be the associated C^*-algebras and let $\mathcal{X}_i$, $i = 1, 2$, be the associated C^*-correspondences. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic, then there is a homeomorphism $f : K^1 \to K^2$, a finite open cover of K^1, $\{U_1, \ldots, U_m\}$, and for each U_j there is a permutation $\sigma_j \in S_n$ ($n = |E|$) such that*

$$f^{-1} \circ \phi_e^2 \circ f|_{U_j} = \phi^1_{\sigma_j(e)}|_{U_j} \text{ for all } e \in E, i \in \{1, \ldots, m\}. \tag{3.1}$$

Proof. Since $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic, there is a C^*-isomorphism $\beta : A_2 \to A_1$ and a C^*-correspondence isomorphism $W : \mathcal{X}_2 \to \mathcal{X}_1$ such that $W(b_1 \cdot \xi \cdot b_2) = \beta(b_1)W(\xi)\beta(b_2)$ and $\langle W(\xi), W(\eta)\rangle_{A_1} = \beta(\langle \xi, \eta \rangle_{A_2})$, for all $b_1, b_2 \in A_2$, $\xi, \eta \in \mathcal{X}_2$. Let $f : K^1 \to K^2$ be the homeomorphism which implements β, that is $\beta(b) = b \circ f$, for all $b \in A_2$.

Let $\delta_e \in \mathcal{X}_2$, defined by

$$\delta_e(g,y) = \begin{cases} 1, & \text{if } e = g \\ 0, & \text{otherwise,} \end{cases}$$

for $e \in E$, be the natural basis in $\mathcal{X}_2$ and let $(\delta'_e)_{e\in E} \subset \mathcal{X}_1$ be the natural basis in $\mathcal{X}_1$, which is defined similarly.

For $\xi \in \mathcal{X}_2$, $\xi = \sum_{g\in E} \delta_g \cdot \xi_g$, where $\xi_g(y) = \xi(g,y)$ for all $y \in K^2_{r(g)}$ and is 0 otherwise. With respect to the bases, we can write

$$\begin{aligned} W(\xi) =& W\left(\sum_{g\in E} \delta_g \cdot \xi_g\right) = \sum_{g\in E} W(\delta_g)\cdot \xi_g \circ f \\ =& \sum_{g\in E}\sum_{e\in E} \delta'_e \cdot w_{eg}\xi_g \circ f, \end{aligned} \tag{3.2}$$

where

$$W(\delta_g) = \sum_{e\in E} \delta'_e \cdot w_{eg} \quad , w_{eg} \in A_1 \tag{3.3}$$

and w_{eg} are given by the formula $w_{eg} = \langle \delta'_e, W(\delta_g)\rangle_{A_1}$, for all $e, g \in E$. We call $(w_{eg})_{e,g\in E}$ the matrix of W with respect to the basis $(\delta'_e)_{e\in E}$ and $(\delta_g)_{g\in E}$ (it is an $n \times n$ matrix, where $n = |E|$). Since W preserves the inner product, we see that

$$\langle W(\delta_g), W(\delta_e)\rangle = \langle \delta_g, \delta_e\rangle = \delta_{ge}, \tag{3.4}$$

where $\delta_{ge}(x) = 1$ if $e = g$ and $x \in K_{r(e)}$ and is 0 otherwise. Also

$$\langle W(\delta_g), W(\delta_e)\rangle = \sum_{f\in E} w^*_{fg} w_{fe}. \tag{3.5}$$

Equations (3.4) and (3.5) imply that for every $x \in K^1$ the matrix $(w_{ef}(x))_{e,f\in E}$ is invertible. Hence there is $\sigma_x \in S_n$ such that $w_{\sigma_x(e)e}(x) \neq 0$ for all $e \in E$. Therefore there is a neighborhood U_x of x such that

$$w_{\sigma_x(e)e}(y) \neq 0 \text{ for all } e \in E, y \in U_x \text{ and } x \in K^1. \tag{3.6}$$

Let $b \in A_2$. Then, for $h \in E$ we have that

$$W(b\cdot\delta_h) = \sum_{e\in E} \delta'_e w_{eh} b \circ \phi^2_h \circ f$$

and

$$\beta(b)\cdot W(\delta_h) = \sum_{e\in E} \delta'_e b \circ f \circ \phi^1_e w_{eh}.$$

Fix $x \in K^1$ and let $\sigma_x \in S_n$ and U_x be defined as in Equation (3.6). Then

$$W(b\cdot\delta_h)(\sigma_x(h), y) = w_{\sigma_x(h)h}(y) b \circ \phi^2_h \circ f(y)$$

and

$$(\beta(b)\cdot W(\delta_h))(\sigma_x(h), y) = b \circ f \circ \phi^1_{\sigma_x(h)}(y) w_{\sigma_x(h)h}(y)$$

for all $y \in U_x$ and for all $h \in E$. Since W is a C^*-correspondence isomorphism and $w_{\sigma_x(h)h}(y) \neq 0$ for all $y \in U_x$, for any $x \in K^1$, there is a neighborhood U_x of x in K^1 and there is a permutation $\sigma_x \in S_n$ such that

$$f^{-1} \circ \phi_h^2 \circ f|_{U_x} = \phi^1_{\sigma_x(h)}|_{U_x} \text{ for all } h \in E.$$

Hence we can find a finite cover $\{U_1, \ldots, U_m\}$ of K^1 and for each U_i we can find a permutation $\sigma_i \in S_n$ such that the Equation (3.1) holds. □

In the special case when the two Mauldin-Williams graphs are totally disconnected, more can be said about the choice of the permutations σ_i.

Corollary 3.4. *Let $\mathcal{G}_i = (G, (K_v^i)_{v\in V}, (\phi_e^i)_{e\in E})$ be two Mauldin-Williams graphs. Let $A_i = C(K^i)$ and let $\mathcal{X}_i$, $i = 1, 2$, be the associated C^*-algebras and C^*-correspondences. If $\mathcal{G}_1$ is totally disconnected and if $\mathcal{X}_1$ is isomorphic with $\mathcal{X}_2$ there is a continuous map $h : K^1 \to S_n$ such that $f^{-1} \circ \phi_e^2 \circ f(x) = \phi_{h(x)(e)}(x)$, for all $x \in K^1$.*

Proof. Recall that if $\mathcal{G}_1$ is totally disconnected, then $\phi_e^1(K_{r(e)}) \bigcap \phi_f^1(K_{r(f)}) = \emptyset$ if $e \neq f$. From the Theorem 3.3 we know that there are open sets $\{U_1, \cdots, U_m\}$, for some $m \in \mathbb{N}$, and permutations $\sigma_1, \cdots, \sigma_m \in S_n$ such that

$$f^{-1} \circ \phi_e^2 \circ f|_{U_i} = \phi^1_{\sigma_i(e)}|_{U_i} \quad \text{for all } e \in E, i \in \{1, \cdots, m\}. \tag{3.7}$$

If $U_i \cap U_j \neq \emptyset$ for some $i \neq j$, then it follows that $\phi^1_{\sigma_i(e)|U_i\cap U_j} = \phi^1_{\sigma_j(e)}|_{U_i\cap U_j}$ for all $e \in E$, hence $\sigma_i(e) = \sigma_j(e)$ for all $e \in E$, so $\sigma_i = \sigma_j$. Therefore we can choose the cover $U_1, \cdots, U_m$ such that $U_i \cap U_j = \emptyset$ if $i \neq j$.

Let $x \in K^1$. Then there is a unique $i \in \{1, \cdots, n\}$ such that $x \in U_i$. We define $h(x) = \sigma_i$. Then $h : K^1 \to S_n$ is a well defined map. Moreover, h is continuous (considering S_n endowed with the discrete topology), since for every $\sigma \in S_n$, $h^{-1}(\sigma) = \emptyset$ or $h^{-1}(\sigma) = U_i$, for some $i \in \{1, \cdots, n\}$. Finally, from the Equation (3.7) we obtain that

$$f^{-1} \circ \phi_e^2 \circ f(x) = \phi^1_{h(x)(e)}(x) \text{ for all } x \in K^1 \text{ and } e \in E.$$

□

Suppose that $\mathcal{G}_i = (G, (K_v^i)_{v\in V}, (\phi_e^i)_{e\in E})$ are two Mauldin-Williams graphs, that satisfy the hypothesis of the Corollary 3.4. We claim that $\mathcal{G}_2$ must also be totally disconnected. Suppose that there are $e, f \in E$, $e \neq f$, such that $\phi_e^2(K^2_{r(e)}) \bigcap \phi_f^2(K^2_{r(f)}) \neq \emptyset$. Then there is an $x \in K^1$ such that $y := f(x) \in \phi_e^2(K^2_{r(e)}) \bigcap \phi_f^2(K^2_{r(f)})$. Then $\phi^1_{h(x)(e)}(x) = \phi^1_{h(x)(f)}(x)$, which is a contradiction since $\mathcal{G}_1$ is totally disconnected. So $\mathcal{G}_2$ is totally disconnected.

Example 3.5. Let K be the Cantor set, let $\phi_i : K \to K$, $i = 1, 2$ be the maps defined by the formulae $\phi_1(x) = \frac{1}{3}x$ and $\phi_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then K is the invariant set of (ϕ_1, ϕ_2). Let $T = [0, 1]$ and let $\psi_i : T \to T$, $i = 1, 2$ be the maps defined by the formulae $\psi_1(x) = \frac{1}{2}x$ and $\psi_2(x) = -\frac{1}{2}x + 1$. Then T is the invariant set of (ψ_1, ψ_2). Since (ψ_1, ψ_2) is not totally disconnected, we see that the associated C^*-correspondences are not strongly Morita equivalent. Hence the tensor algebras fail to be strongly Morita equivalent in the sense of [2], yet their C^*-envelopes coincide with $\mathcal{O}_2$.

Example 3.6. Let T be the regular triangle in $\mathbb{R}^2$ with vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (1/2, \sqrt{3}/2)$. Let $\phi_1(x, y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right)$, $\phi_2(x, y) = \left(\frac{x}{2}, \frac{y}{2}\right)$ and $\phi_3(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right)$. Then the invariant set K of this iterated function system is the Sierpinski gasket. Let $\psi_1 = \sigma_1 \circ \phi_1$,

$\psi_2 = \phi_2$ and $\psi_3 = \sigma_3 \circ \phi_3$, where σ_i is the symmetry about the median from the vertex $\phi_i(C)$ of the triangle $\phi_i(T)$, for $i = 1, 3$. Then the invariant set of this iterated function system is also the Sierpinski gasket, but the C^*-correspondences associated to (ϕ_1, ϕ_2, ϕ_3) and (ψ_1, ψ_2, ψ_3) are not isomorphic.

References

[1] M. F. Barnsley, *Fractals everywhere*, Second edition, Academic Press Professional, Boston, MA, 1993.

[2] D. P. Blecher,P. S. Muhly, V. I. Paulsen, 'Categories of operator modules (Morita equivalence and projective modules).' Mem. Amer. Math. Soc. 143 (2000), no. 681, viii+94 pp.

[3] J. Cuntz, 'Simple C^*-algebras generated by isometries', *Comm. Math. Phys*, 57 (1977), no. 2, 173–185.

[4] J. Cuntz and W. Krieger, 'A class of C^*-algebras and topological Markov chains', *Invent. Math.* 56 (1980), no. 3, 251–268.

[5] G. A. Edgar, *Measure, topology, and fractal geometry*, Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1990.

[6] N. J. Fowler, P. S. Muhly and I. Raeburn, 'Representations of Cuntz-Pimsner Algebras', *Indiana Univ. Math. J.* 52 (2003), no. 3, 569–605.

[7] J. E. Hutchinson, 'Fractals and self-similarity', *Indiana Univ. Math. J.* 30 (1981), no. 5, 713–747.

[8] M. Ionescu, 'Operator Algebras and Mauldin-Williams graphs', preprint OA/0401408.

[9] T. Kajiwara and Y. Watatani, 'C^*-algebras associated with self-similar sets', preprint OA/0312481.

[10] T. Kajiwara and Y. Watatani, 'KMS states on C^*-algebras associated with self-similar sets', preprint OA/0405514.

[11] J. Kigami, *Analysis on Fractals*, 2001, Cambridge University Press.

[12] A. Kumjian, D. Pask and I. Raeburn, 'Cuntz-Krieger algebras of directed graphs', *Pacific J. Math.* 184 (1998), no. 1, 161–174.

[13] E. C. Lance, *Hilbert C^*-modules. A toolkit for operator algebraists*, London Mathematical Society Lecture Note Series, 210. Cambridge University Press, Cambridge, 1995.

[14] R. D. Mauldin and S. C. Williams, 'Hausdorff dimension in graph directed constructions', *Trans. Amer. Math. Soc.* 309 (1988), no. 2, 811–829.

[15] P. S. Muhly and B. Solel, 'Tensor algebras over C^*-correspondences: representations, dilations, and C^*-envelopes', *J. Funct. Anal.* 158 (1998), no. 2, 389–457.

[16] P. S. Muhly and B. Solel, 'On the Morita equivalence of tensor algebras', *Proc. London Math. Soc.* (3) 81 (2000), no. 1, 113–168.

[17] M. V. Pimsner, 'A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$', *Free probability theory* (Waterloo, ON, 1995), 189–212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997.

[18] C. Pinzari, Y. Watatani and K. Yonetani, 'KMS states, entropy and the variational principle in full C^*-dynamical systems', *Comm. Math. Phys.* 213 (2000), no. 2, 331–379.

[19] I. Raeburn, 'On the Picard group of a continuous trace C^*-algebra', *Trans. Amer. Math. Soc.* 263 (1981), no. 1, 183–205.

[20] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, 60. American Mathematical Society, Providence, RI, 1998.

[21] M. A. Rieffel, 'Induced representations of C^*-algebras', *Advances in Math. 13 (1974), 176–257.*

[22] M. A. Rieffel, 'Morita equivalence for C^*-algebras and W^*-algebras', *J. Pure Appl. Algebra 5 (1974), 51–96.*